Let $F(x_1, x_2)$ be the cdf of $X_1, X_2$.

$$\max [F_1(x_1) + F_2(x_2) - 1, 0] \leq F(x_1, x_2) \leq \min [F_1(x_1), F_2(x_2)]$$

where

$F_1(x_1)$ is the marginal distribution of $X_1$

$F_2(x_2)$ is the marginal distribution of $X_2$

Upper Limit: Obvious
Clearly

\[ F(x_1, x_2) \leq F_1(x_1) \]
\[ F(x_1, x_2) \leq F_2(x_2) \]
\[ F(x_1, x_2) \leq \min (F_1(x_1), F_2(x_2)) \]

The upper bound is attained at \( F_1(x_1) = F_2(x_2) \)

Assuming absolute continuity.

Best Ranked with Best, \( x_1 = F_1^{-1} F_2(x_2) \)

Lower Bound: \( F(x_1, x_2) \geq 0 \)
\[ P(A) + P(D) + P(C) + P(B) = 1 \]
\[ P(A) \geq 0 \quad P(B) \geq 0 \quad P(C) \geq 0 \quad P(D) \geq 0 \]

\[
\begin{align*}
P(A) + P(D) & = F_1(x_1) \\
P(B) + P(D) & = F_2(x_2) \\
P(D) & = F(x_1, x_2)
\end{align*}
\]

\[
\therefore F_1(x_1) + F_2(x_2) - F(x_1, x_2) + P(C) = 1
\]

\[
\therefore F(x_1, x_2) = F_1(x_1) + F_2(x_2) - 1 + P(C)
\]

\[ P(C) \geq 0 \]

\[
\therefore F(x_1, x_2) \geq F_1(x_1) + F_2(x_2) - 1
\]

Lower bound is attained at \( F_1(x_1) = 1 - F_2(x_2) \) best in one with worst in other.
Can We Improve on These Bounds By Using The Information On Maximization?  
(See Peterson, 1976)

\[ F(x_1 \leq x_1, x_2 \leq x_2) \]

\[ = \Pr(X_1 \leq x_1, X_2 \leq x_2, X_1 < X_2) + \Pr(X_1 \leq x_1, X_2 \leq x_2, X_1 > X_2) \]

(Assume \( \Pr(X_1 = X_2) = 0 \); no ties. Distribution is absolutely continuous)
Now
\[ \Pr(X_1 \leq x_1, X_2 \leq x_2, X_1 < X_2) \leq \Pr(X_2 \leq x_2, X_1 < X_2) = F_2(x_2, X_2 > X_1) \]
This is our data.

\[ \Pr(X_1 \leq x_1, X_2 \leq x_2, X_1 > X_2) \leq \Pr(X_1 \leq x_1, X_1 > X_2) = F_1(x_1, X_1 > X_2) \]
This is our data.

\[ \therefore \text{Upper bound is} \]
\[ F(X_1 \leq x_1, X_2 \leq x_2) \leq F_2(x_2, X_2 > X_1) + F_1(x_1, X_1 > X_2) \]
(Will Show it is tight)

\[ F_2(x_2, X_2 > X_1) \leq F_2(x_2) \quad ? \text{Yes, obviously} \]
\[ F_1(x_1, X_1 > X_2) \leq F_1(x_1) \quad ? \text{Yes, obviously} \]
But not necessarily true that

\[ F_2(x_2, X_2 > X_1) + F_1(x_1, X_1 > X_2) \leq \min [F_1(x_1), F_2(x_2)]. \]

For this to be true, we would require that if \( F_1(x_1) \leq F_2(x_2) \),

\[ F_2(x_2, X_2 > X_1) + F_1(x_1, X_1 > X_2) < F_1(x_1, X_1 > X_2) + F_1(x_1, X_2 > X_1) \]

so

\[ F_2(x_2, X_2 > X_1) < F_1(x_1, X_2 > X_1). \]

But this cannot always hold. Suppose that \( X_2 \) and \( X_1 \) are exchangeable. We can reverse inequality depending on sign of \( x_1, x_2 \).
Lower Bound

\[ \Pr (X_1 \leq x_1, X_2 \leq x_2, X_1 > X_2) \geq \Pr (X_1 \leq \min(x_1, x_2), X_2 \leq \min(x_1, x_2), X_1 > X_2) \]

\[ = \Pr (X_1 \leq \min(x_1, x_2), X_1 > X_2) \]

\[ = F_1(\min(x_1, x_2), X_1 > X_2) \]

\[ \Pr (X_1 \leq x_1, X_2 \leq x_2, X_1 < X_2) \geq \Pr (X_1 \leq \min(x_1, x_2), X_2 \leq \min(x_1, x_2), X_1 < X_2) \]

\[ = \Pr (X_1 \leq \min(x_1, x_2), X_1 < X_2) \]

\[ = F_2(\min(x_1, x_2), X_1 < X_2) \]

\[ F_1(\min(x_1, x_2), X_1 > X_2) + F_2(\min(x_1, x_2), X_1 < X_2) \leq F(x_1, x_2) \]

Obviously greater than or equal to 0. But why is

\[ F_1 (x_1) + F_2 (x_2) - 1 \leq F_1 (\min (x_1, x_2), X_1 > X_2) + F_2 (\min (x_1, x_2), X_2 > X_1) \]

Can we order relative to the Frechet lower bound?
If we consider the set
\[ \{ (x_1, x_2) : F_2(x_2) \leq 1 - F_1(x_1) \} \]
which under our strict monotonicity assumption is
\[ \{ (x_1, x_2) : x_2 \leq F_2^{-1} (1 - F_1(x_1)) \} . \]
Thus in this set
\[ F_1(x_1) + F_2(x_2) - 1 \leq 0. \]
Clearly in this set, the Frechet lower bound is 0. And the right hand side is non-negative.
So the Peterson bound is larger (no smaller) than the Frechet bound over this region. What about the other region?
For $x_1 = x_2 = x$

\[
\Pr(X_1 \leq x_1) = F_1(x_1) = P(A) + P(B)
\]
\[
\Pr(X_2 \leq x_2) = F_2(x_2) = P(D) + P(B)
\]

Now

\[
F_1(x, X_1 > X_2) + F_2(x, X_1 < X_2)
= P(B)
\]

Then clearly

\[
P(A) + P(B) + P(D) + P(B) \leq P(B) + \underbrace{P(A) + P(B) + P(C) + P(D)}_{1}
\]

\therefore \text{ inequality follows. } (P(C) \geq 0)
\[ F(x) = 1 - F(x) \]

\[ x_2 = x_1 \]

\[ F_2(x_2) = 1 - F_1(x_1) \]
For this to be true, generally, we need if $x_1 < x_2$

$$0 \leq F_1(x_1) + F_2(x_2) - 1 \leq F_1(x_1, X_1 > X_2) + F_2(x_1, X_1 < X_2)$$

which is equivalent to

$$1 \leq F_1(x_1) + F_2(x_2) \leq 1 + F_1(x_1, X_1 > X_2) + F_2(x_1, X_1 < X_2)$$

See the next figure:
Now suppose $x_2 \geq x_1$

$$F_2(x_2) = 1 - F_1(x_1)$$

$$(x_2) = F_2^{-1}(1 - F_1(x_1))$$
Now we can do the same type of decomposition

Now

\[ A = A' \cup A'', \quad A' \cap A'' = \emptyset \]
\[ C = C' \cup C'', \quad C' \cap C'' = \emptyset \]

(where \( x_1 = x \) in the previous figure)

We are on the upper side of the boundary when \( x_1 \leq x_2 \).

\[
F_1(x_1) = P(B) + P(A') + P(A'') \\
F_2(x_1) = P(B) + P(D) \\
F_2(x_2) = P(B) + P(A') + P(C') + P(D)
\]

\[
P(B) + P(A') + P(A'') + P(B) + P(A') + P(C') + P(D) \\
\leq P(B) + P(A') + P(A'') + P(C') + P(C'') + P(D) + P(B)
\]

which is obviously true if \( P(C'') \geq P(D) \). But in general this is not true.
Are Bounds Sharp?

Need to show for each bound there exists a joint distribution having specified marginals that has probability mass arbitrarily close to the mass of the bound.

Lower Bound: Construct A Lower Bound Joint

Lower Bound for distribution.

\[
F_i(x_1, x_2) \text{ lower bound} \\
= F_1(x_1, X_1 > X_2) \text{ [evaluated at } (X_1, X_2) = (s, s + \varepsilon)] \\
+ F_2(x_2, X_1 < X_2) \text{ [evaluated at } (X_1, X_2) = (s + \varepsilon, s)]
\]

Let \( \varepsilon > 0 \) be arbitrarily small, and we attain lower bound.
Upper Bound for the distribution

\[ F_u(x_1, x_2) \text{ upper bound} \]
\[ = F_1(x_1, X_1 > X_2) \quad [\text{evaluated at } (X_1, X_2) = (s + \eta, s)] \]
\[ + F_2(x_2, X_1 < X_2) \quad [\text{evaluated at } (X_1, X_2) = (s, s + \eta)] \]

Let \( \eta \) be arbitrarily large, and we get a cdf (arbitrarily close to cdf).

\[ \therefore \text{ Bounds tight.} \]

This proof uses the same idea as in our Roy proof.
First Term
Second Term
As $\varepsilon \to 0$ we get a joint distribution. Upper Bound works by setting $\varepsilon \to \infty$.

Observe several features of the Peterson bounds.

(1) Bounds collapse when $x_1 = x_2$.

(2) In general, bounds do not collapse.

(3) Lower bound is a joint pdf that places all of its weight on the line $x_1 = x_2$.

(4) Upper bound is a joint pdf placing all of its weight on $(s, \infty)$ and $(\infty, s)$.
Consider More General Setup

\[ Y_1, Y_2, I \quad \text{(generalized Roy)} \]

\[ D = 1 (I > 0) \]
\[ 1 - D = 1 (I < 0) \]

We observe

\[ F(Y_1 | I < 0) = F(Y_1 | D = 0) \]
\[ F(Y_2 | I > 0) = F(Y_2 | D = 1) \]
\[ F(Y_1, Y_2) \leq \min[F_1(Y_1), F_2(Y_2)] \]

We can identify \( I \) (to be shown)

\[ F(Y_1, Y_2 | I) \leq \min[F_1(Y_1 | I), F_2(Y_2 | I)] \quad \text{(pointwise Frechet)} \]

\[ \int F(Y_1, Y_2 | I) d\mu(I) \leq \int \min[F_1(Y_1 | I), F_2(Y_2 | I)] d\mu(I) \]
\[ \leq \min[F_1(Y_1), F_2(Y_2)] \]

\( \mu(I) \) is the probability distribution of \( I \).
**Lower Bound**

Frechet is

\[ F (Y_1, Y_2) \geq \max [0, \ F_1 (Y_1) + F_2 (Y_2) - 1] \]

Now for each \( I \) we have that

\[ F (Y_1, Y_2 \mid I) \geq \max [0, \ F_1 (Y_1 \mid I) + F_2 (Y_2 \mid I) - 1] \]

Question. Is

\[ \int \max [0, \ F_1 (Y_1 \mid I) + F_2 (Y_2 \mid I) - 1] \, d\mu (I) \geq \max [0, \ F_1 (Y_1) + F_2 (Y_2) - 1] ? \]

Yes, trivially

\[ \int \max [A (I), B (I)] \, d\mu (I) \geq \max \left[ \int A (I) \, d\mu (I), \ \int B (I) \, d\mu (I) \right] \]