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The Assignment Game I: The Core

By L. S. SHAPLEY and M. SHUBIK

Abstract: The assignment game is a model for a two-sided market in which a product that comes in large, indivisible units (e.g., houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit. The units need not be alike, and the same unit may have different values to different participants. It is shown here that the outcomes in the core of such a game — i.e., those that cannot be improved upon by any subset of players — are the solutions of a certain linear programming problem dual to the optimal assignment problem, and that these outcomes correspond exactly to the price-lists that competitively balance supply and demand. The geometric structure of the core is then described and interpreted in economic terms, with explicit attention given to the special case (familiar in the classic literature) in which there is no product differentiation — i.e., in which the units are interchangeable. Finally, a critique of the core solution reveals an insensitivity to some of the bargaining possibilities inherent in the situation, and indicates that further analysis would be desirable using other game-theoretic solution concepts.

1. Introduction

1.1. Two-Sided Market Games

Two-sided market models are important, as Cournos, Edgeworth, Böhm-Bawerk, and others have observed, not only for the insights they may give into more general economic situations with many types of traders, consumers, and producers, but also for the simple reason that in real life many markets and most actual transactions are in fact bilateral — i.e., bring together a buyer and a seller of a single commodity. Modern game-theoretic concepts, when applied to even the most elementary economic models, have often yielded suggestive results, sometimes reinforcing and sometimes challenging the more traditional doctrines based on behavioristic theories of the individual. The present study, of which this paper is the first part, will concern a class of simple, two-sided market "games" whose distinctive feature is the indivisibility and the ability to satiate of the goods for sale, e.g., houses or automobiles, so that the primary object of the game is simply to find suitable "assignments" of buyers to sellers.

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2) LLOYD S. SHAPLEY, The Rand Corporation, Santa Monica, California, USA.
3) MARTIN SHUBIK, Yale University, New Haven, Connecticut.
4) See references at the end of this paper.
5) See e.g., DERNER and SCARF; SHAPLEY 1959; SHAPLEY and SHUBIK 1959; SHAPLEY and SHUBIK W. Y.; SHUBIK.
6) This model was first treated by SHAPLEY in 1955; the present account has been extracted from the manuscript of a book in preparation. A short, non-game-theoretic account will be found in Gale [pp. 160–162].
We intend to explore the properties of such assignment games from several different solutional viewpoints. In this first part we shall concentrate on the core of the game — i.e., the set of outcomes that no coalition can improve upon.

1.2. The Underlying Economic Assumptions

The assumptions of our model, though restrictive in many respects, do permit considerable latitude in size and structure. There may be many or few traders on either side of the market, there may be product differentiation, and the traders themselves may be quite dissimilar in their likes and dislikes. Thus, a wide range of specific models, from a situation where, say, two or three firms are bidding for the same building site, to a large "noisy" milieu like the private-party market in used cars, are encompassed within the same framework and can be given a more or less uniform treatment.

Let us list the main economic assumptions:
1. Utility is identified with money.
2. Side payments are permitted.
3. The objects of trade are indivisible.
4. Supply and demand functions are inflexible.

This is not the place for extended comment on the first assumption. We do, however, stress one point: In the open marketplace, where there are many individuals buying and selling for money (often acting as fiduciaries or trustees — i.e., using other people's money), the full monetization of utility comes closest to a practical realization. Here if anywhere the assumption of "u-money" is defensible.

The second assumption is largely one of convenience in modeling and analysis. By permitting free transfer of money among all participants we avoid necessity of providing by special rules for the ordinary payments from a customer to his supplier. To be sure, we are also permitting payments to third parties, i.e., "side payments" in ordinary parlance. But it is worth noting that the avoidance of such payments, in situations where they are in fact avoided, is generally more a matter of custom or ethics than of hard and fast rules. The core solution, as it turns out, excludes third-party payments. We shall see latter that the value solution usually requires third-party payments, for reasons related to the discussion in Sec. 5 below.

The stable-set solutions, which are interpreted as "standards of behavior" in the von Neumann-Morgenstern theory, have it both ways: some standards of behavior allow third-party payments while others do not.

The third assumption is rather unusual, since it is more often the opposite hypothesis (i.e., fungibility and perfect divisibility) that is imposed in the name of mathematical simplification. Indivisibility is usually considered an "imperfection" in the market — a technical and conceptual difficulty. Indeed, our present approach leads us towards combinatorics and linear programming, and away from the differential calculus methods that pervade so much of traditional economics. There is no dearth of economic applications under this assumption, however; a classic example is BöhM-Bawerks horse market (see Sec. 4).

Inflexibility, the fourth assumption, in a way includes the third. Having postulated indivisible commodities, we further assume that each producer has a supply of exactly one item, and each consumer a desire for exactly one item. One might achieve much the same effect in terms of perfectly divisible goods by assuming individual supply and demand functions of a step-function type, as illustrated below for the case of undifferentiated products.

Our assumptions are admittedly restrictive, but they are not entirely unrealistic and we are not seeking wide generality for our model in any case. While excluding many important but secondary considerations, we wish to focus attention on what we consider the basic motive force of competitive economics, namely the profitable interaction between separately maximizing individuals.

2. Description of the Game
2.1. A Real Estate Market

We shall now formulate the market game in detail, motivating it in terms of a market in private homes. Another interpretation, less specific and hence more versatile, will be outlined in Sec. 2.3.

Let there be \(m\) homeowners in the market, and \(n\) prospective purchasers. We shall refer to them simply as sellers and buyers, respectively. The \(i^{\text{th}}\) seller values his house at \(c_i\) dollars, while the \(j^{\text{th}}\) buyer values the same house at \(h_{ij}\) dollars.

---

1) For treatments of other solution concepts in certain special cases of the assignment game see Shapley [1955 and 1961] and Shapley and Shubik [1969], also von Neumann and Morgenstern (pp. 555 - 596).
2) See for example Shapley and Shubik [1966, pp. 807 - 808].
3) See also Shapley [1961], Shapley and Shubik [1969].
5) For a modern treatment of the problem of indivisibilities in a competitive market, see Henry.
6) For a numerical example, see Table 1, in Sec. 3.4.
A moment's reflection reveals that these mixed pairs are the only essential coalitions in the game. The best that a larger coalition can do is to split up into separate trading pairs and pool the profit. Hence the $m \times n$ matrix $(a_{ij})$ suffices to determine $v$ completely. In fact, $v$ may be characterized as the smallest super-additive set function on $M \cup N$ satisfying (2.3) and (2.5).

In order actually to compute $v$ for the larger mixed coalitions, we must pick out an optimal set of transactions, maximizing the coalition's total gain. Put symbolically, we have

$$v(S) = \max \left[ a_{i_1j_1} + a_{i_2j_2} + \cdots + a_{i_kj_k} \right],$$

the maximum to be taken over all arrangements of $k$ distinct players $i_1, \ldots, i_k$ in $S \cap M$ and $j_1, \ldots, j_k$ in $S \cap N$, where $k = \min(|S \cap M|, |S \cap N|)$. The evaluation of an expression such as (2.6) is commonly called the "optimal assignment problem" or simply "assignment problem"; accordingly we refer to games of this form as assignment games.

For relatively small coalitions $S$ (say, less than 10 players) the maximum in (2.6) can often be discovered by inspection; the reader may try his hand on the examples below. (In the first three are four optimal assignments, in the second there is just one; the respective values are 4 and 16.)

<table>
<thead>
<tr>
<th>$S \cap N$</th>
<th>$S \cap N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S \cap M$</td>
<td>$S \cap M$</td>
</tr>
<tr>
<td>0 2 0</td>
<td>5 8 2</td>
</tr>
<tr>
<td>2 0 2</td>
<td>7 9 6</td>
</tr>
<tr>
<td>0 2 0</td>
<td>3 0</td>
</tr>
</tbody>
</table>

For larger matrices, systematic methods are available for finding the optimal assignment.

For the most part we shall be interested in evaluating (2.6) only for the all-player coalition $S = M \cup N$. The number $v(M \cup N)$ is obviously very important, since it determines the maximum total monetary payoff, and hence the Pareto set and imputation space of the game. The optimal assignments for the other large, nonessential coalitions have little effect on the analysis of the game.

2.3. A Game of Monogamy

It will be observed that the characteristic function of the assignment game, when defined in terms of the numbers $a_{ij}$, is symmetric in its treatment of buyers and sellers, although the market model itself was not. The following alternative

---

1) A coalition is "essential" if it has more than one member and exhibits strict superadditivity for every split-up into smaller coalitions, i.e., $v(S) > v(S_1) + v(S_2)$ for all $S_1, S_2$ with $S_1 \neq \emptyset, S_2 \neq \emptyset, S_1 \cup S_2 = S$, and $S_1 \cap S_2 = \emptyset$.

2) See e.g., DANTZIG [Chapter 15].
interpretation of the game, in sidestepping the preliminaries that went into the "real estate" version, has the conceptual advantage of being symmetrical from the start: 1)

We may imagine an economic milieu in which two types of agents, say "producers" and "consumers" (or "men" and "women"), are constrained for some reason to conduct their business under exclusive, bilateral contracts. Thus, after an initial period of jockeying for position, a number of partnerships are formed, and any transfers of goods or services (but not money) are limited to exchanges between partners. In this model we do not care particularly about the nature of the goods or services; we only care that with every partnership if there is associated a number $a_{ij} \geq 0$, denoting the potential for profit of that partnership if it forms. If we now regard the system as a cooperative game, it is apparent that its characteristic function is given by (2.4) and (2.6).

A contract between players $i$ and $j$ would of course specify how the gain $a_{ij}$ is to be divided. A prudent, "economic" man playing this game would be loath to enter a partnership for a stated share of the proceeds until he had satisfied himself that more favorable terms could not be obtained elsewhere. We can imagine that each player would set a price on his participation, and that no contracts would be signed until the prices on both sides of each partnership formed are in harmony, i.e., satisfy each partner and add up to what the partnership is worth. One may well ask: does a set of such "harmonious" prices exist?

Mathematically, this question may be put as follows: Do there exist numbers $q_i \geq 0, r_j \geq 0$ such that

\[
\begin{align*}
q_i + r_j & = a_{ij} \quad \text{if } i \text{ and } j \text{ are ultimately partners}, \\
q_i + r_j & = a_{ij} \quad \text{if } i \text{ and } j \text{ are not partners}.
\end{align*}
\]

The answer is in the affirmative, as we shall presently demonstrate. This fact is of central importance throughout the subsequent analysis of the assignment game.

2.4. Complementarity

Before proceeding to a study of the solutions of the assignment game, we shall digress briefly to describe an interesting property of the characteristic function.

The superadditivity of $v$ in (2.6) is obvious, since combining two coalitions only enlarges the set of possible assignments. Less obvious is the way in which the marginal value of a player responds to changes in the make-up of the coalition to which he belongs. Intuitively, we should expect a player's value to a coalition to be small when there is an excess of players of his type, and large when there is an excess of players of the other type. In other words, similar players should appear somewhat as substitutes in coalitional matters, and dissimilar players as complements. The following theorem gives an expression to this intuitive idea.

A special notation will be useful: $S'$ will denote the coalition consisting of $S$ and the added player $p$. (We shall use this notation only when $p$ is not a member of $S$.) Then the marginal value of player $p$ to the coalition $S$ is given by

\[
m(p, S) = v(S') - v(S).
\]

The following result is proved in Shapley [1962].

**Theorem 1:**

If $p$ and $q$ are of the same type (i.e., both in $M$ or both in $N$), then

\[
m(p, S) \leq m(p, S), \quad \text{all } S \ni p, q.
\]

If they are of the opposite type, then

\[
m(p, S') \geq m(p, S), \quad \text{all } S \ni p, q.
\]

It will be seen that this theorem is simply a statement about the sign of the "second difference" $v(S') - v(S') - v(S') + v(S)$. When this quantity is positive, the players are complementary inputs to the coalition-forming process; when it is negative, they are substitutes.

3. The Core

3.1. LP Form of the Model

It will be useful at this point to recast the assignment problem (2.6) into linear programming (LP) terminology.

Consider just the assignment problem for the coalition of all players — i.e., the problem of determining $v(M \cup N)$. Introduce $m n$ nonnegative real variables $x_{ij}$, $i \in M, j \in N$, and impose on them the $m + n$ constraints

\[
\sum_{i \in M} x_{ij} \leq 1, \quad \sum_{j \in N} x_{ij} \leq 1.
\]

(We may interpret $x_{ij}$ in the real estate game as the fraction of the $i$th house sold to player $j$, or, in the partnership game, as the probability that partnership $ij$ will form.) The LP problem is then to maximize the following objective function:

\[
z = \sum_{i \in M, j \in N} a_{ij} x_{ij}.
\]

It can be shown [see Dantzig, p. 318] that the maximum value $z_{\text{max}}$ is attained \footnote{Cf. Gale and Shapley [1962].} with all $x_{ij} = 0$ or 1. Thus, the fractions or probabilities artificially introduced disappear from the solution, and the (continuous) LP problem is effectively equivalent to the (discrete) assignment problem, so that we have

\[
z_{\text{max}} = v(M \cup N).
\]

As is well known, every LP problem can be transposed into a dual form, and the solutions of the two problems are intimately bound up in each other. In the case...
case, the dual has $m + n$ nonnegative real variables, $u_1, \ldots, u_m, v_1, \ldots, v_n$, subject to the $mn$ constraints,
\[
    u_i + v_j \geq a_{ij}, \quad i \in M, \quad j \in N,
\]
and the objective is to minimize the sum:
\[
    w = \sum_{i \in M} u_i + \sum_{j \in N} v_j.
\]

The fundamental duality theorem\(^1\) tells us that $w_{\text{min}} = z_{\text{max}}$.\(^2\)

What meaning does the dual problem have in the present context? Let $(u, v) = (u_1, \ldots, u_m, v_1, \ldots, v_n)$ be a vector that minimizes (3.4), subject to (3.3). Then we have
\[
    \sum_{i \in M} u_i + \sum_{j \in N} v_j = w_{\text{min}} = z_{\text{max}} = v(M \cup N).
\]

This means that $(u, v)$ is an imputation\(^3\) of the assignment game. Moreover, (3.3) tells us that for every pair $i \in M, j \in N$,
\[
    u_i + v_j \geq a_{ij} = v(ij).
\]

It follows, by (2.6), that for any coalition $S$\(^4\),
\[
    \sum_{i \in S \cap M} u_i + \sum_{j \in S \cap N} v_j \geq v(S).
\]

But (3.5) and (3.6) are exactly how the core of the game is defined: (3.5) ensures the feasibility of $(u, v)$ and (3.6) ensures its non-improvability by any coalition. Conversely, any payoff vector in the core — i.e., satisfying (3.5) and (3.6) — clearly fulfills the conditions for a solution to the dual LP problem. Hence we conclude:

**Theorem 2:**

In core of an assignment game is precisely the set of solutions of the LP dual of the corresponding assignment problem.

### 3.2. Prices

Our venture into the field of linear programming has proved most fruitful. We have already garnered (1) a proof of the existence of the core, (2) a characterization of the points in the core, and (3) an assurance of effective computational procedures for both the characteristic function and the core.\(^5\)

\(^1\) See e.g., DANTZIG [p. 129]. The existence of feasible solutions to both primal and dual is apparent from an inspection of (3.1) and (3.3).

\(^2\) I.e., a nonnegative, Pareto optimal payoff vector.

\(^3\) It is interesting that the core can be found without solving the multitude of assignment problems arising from the proper submatrices of $(a_{ij} | i \in M, j \in N)$. Because of this, the core in a large assignment game may be easier to compute than the characteristic function itself. The explanation of the paradox lies in the relatively small number of coalitions that are essential.

\(^4\) The first matrix shown in Sec. 2.2 provides an example: the core of the associated 6-person game contains just the imputation $(0, 2, 0; 0, 2, 0)$. The other matrix shown there is more typical, however.
In general, if numbers \( a_{ij} \geq 0 \) are chosen "at random", the optimal assignment is unique. This implies that a dual solution \((u, v)\) exists in which the strict inequality:

\[ u_i + v_j > a_{ij} \]

holds for all pairs \((i, j)\) that are not involved in the optimal assignment. Let \( i_j \) be one of the \( \min (m, n) \) pairs in the optimal assignment. Then a small amount can be shifted from \( u_i \) to \( v_j \) without spoiling any of the conditions for a dual solution. It follows that there are at least \( \min (m, n) \) "degrees of freedom" in the core. On the other hand, the dimensionality of the core is certainly never greater than \( \min (m, n) \), because if the \( u \)-components of a core vector are given, then the \( v \)-components are completely determined, and vice versa.

A picture of the core is beginning to take shape: it is a closed, convex polyhedral set whose dimension is typically equal to \( \min (m, n) \), but may be less in the presence of degeneracies — i.e., special arithmetical relations among the \( a_{ij} \). Note that the dimension of the imputation space, in which the core is situated, is \( m + n - 1 \), which is considerably larger than \( \min (m, n) \).

We next show that the core tends to be elongated, with its long axis oriented in the direction of market-wide price trends. There is a "high-price corner" of the core, at which every seller gets his top profit and every buyer his bottom. There is also a "low-price corner", where the reverse is true. These points are poles of a diameter; they prove to be at least as far apart as any other two points in the core. Thus, to a considerable extent, the fortunes of all players of the same type rise and fall together.

Let us try to make this intuitively plausible. For simplicity assume \( m = n \). Suppose that we have a core vector at which all players show a positive profit. Then a small constant can be subtracted from all of the sales prices without upsetting the core conditions. Each buyer-seller pair makes the same profit as before, and no individual is forced to take a loss. This price-cutting can continue until some seller is priced out of the market, his profit going to zero. The process can of course be reversed, raising all prices by equal amounts until one of the buyers is driven out. Thus there is a natural "degree of freedom" within the core that corresponds to market-wide price movement.

This phenomenon is often encountered in the cooperative solutions to market games. The heuristic principle may be stated as follows: intergroup allocations are relatively indeterminate, intragroup allocations are relatively precise.

**Theorem 3:**

Over all imputations in the core, let \( u^* \) and \( u_{ai} \) denote the highest and lowest lowest payoffs, respectively, to player \( i \in M \); similarly define \( v^* \) and \( v_{aj} \) for player \( j \in N \). Then the payoff vectors

\[
(u_*, v^*) \quad \text{and} \quad (u^*, v_*)
\]

are themselves in the core. Moreover, no two imputations in the core are further apart than these two.

Figure 3, in § 3.4 below, may help the reader to visualize this theorem. The heart of the proof is in the following lemma:

**Lemma:**

Let \((u', v')\) and \((u^*, v^*)\) be any two imputations in the core. Define

\[
u_i = \min (u'_i, u^*_i), \quad v_j = \min (v'_j, v^*_j),
\]

\[u_i = \max (u'_i, u^*_i), \quad v_j = \max (v'_j, v^*_j).
\]

Then the vectors \((u, v)\) and \((\bar{u}, \bar{v})\) are in the core.

**Proof:**

For any \( i, j \) we have

\[
\sum_{a \in M} u_i + \sum_{b \in N} v_j = \min (u'_i + v'_j, u^*_i + v^*_j)
\]

\[
\geq \min (u'_i + v'_j, u^*_i + v^*_j)
\]

\[\geq a_{ij},
\]

using (3.3). Hence \((u, v)\) is a dominated core. It remains to show that it is an imputation. It is obviously nonnegative; we must therefore show only that its components add up to \( v(M \cup N) \). For convenience, label the players so that the pairs \( \Pi, \Sigma, \ldots, \Omega \) describe an optimal assignment. (Here \( k \leq \min (m, n) \).) Then:

\[
u_i = \min (u'_i, u^*_i) = \min (u_{ai} - v'_i, a_{ii} - v^*_i)
\]

\[= a_{ii} - \max (v'_i, v^*_i) = a_{ii} - v_j.
\]

for \( i \leq k \). Also, for \( i, j > k \) (if any) we have \( u_i = v_j = 0 \). Hence

\[\sum_{i \in M} u_i + \sum_{j \in N} v_j = \sum_{i=1}^{k} a_{ii} = v(M \cup N).
\]

This completes the proof of the lemma.

---

1) If the original data of the real estate version of the game are chosen "at random" — more precisely, from continuous probability distributions that give zero weight to any particular numerical value — then the value 0 may be favored over other values in the assignment matrix, and nonuniqueness in the optimal assignment may occur with positive probability. But this merely reflects indeterminacy in the choice of inactive buyer-seller pairs, involving players who are actually "priced out of the market". The \( u_i \) or \( v_j \) for such players are identically zero in the core, and the dimension of the core is reduced accordingly.

2) This will be illustrated in Sec. 3.4. If \( m \neq n \) a similar discussion applies, except that it may no longer be possible to drive any of the players of the less numerous type down to zero.
To prove the theorem, simply take a finite collection of core vectors that includes all the extreme values going into the definitions of \( u'_i, u_{ai}, v'_j, v_{aj} \), and apply the lemma repeatedly to construct additional core vectors, until \((u^*, v^*)\) are reached. For the last statement of the theorem, note that any two points \((u', v'), (u^*, v^*)\) in the core necessarily satisfy the inequalities:

\[
|u'_i - u^*| \leq u^*_i - u_{ai}, \quad \text{all } i \in M,
\]

and

\[
|v'_j - v^*| \leq v^*_j - v_{aj}, \quad \text{all } j \in N.
\]

Thus, the stated result holds not only for an Euclidean distance but for any distance function based on a linear norm, i.e., that depends only upon the absolute values of the coordinate differences.

### 3.4. A Numerical Example

Let there be three sellers (1, 2, 3) and three buyers (1', 2', 3'), as follows:

<table>
<thead>
<tr>
<th>Houses (i)</th>
<th>Sellers' basis (ci)</th>
<th>Buyers' valuations</th>
<th>(h1)</th>
<th>(h2)</th>
<th>(h3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$18,000</td>
<td>$23,000</td>
<td>$26,000</td>
<td>$20,000</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15,000</td>
<td>22,000</td>
<td>24,000</td>
<td>21,000</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>19,000</td>
<td>21,000</td>
<td>22,000</td>
<td>17,000</td>
<td></td>
</tr>
</tbody>
</table>

These data lead at once to the following \((a_{ij})\) matrix, already considered in Sec. 2.2:

\[
\text{(buyers N)}
\]

\[1~ 2'~ 3'~ u:\]

\[\begin{array}{ccc}
1 & 2 & 3 \\
5 & 2 & 4 \\
\end{array}
\]

\[\text{(sellers M)}
\]

\[\begin{array}{ccc}
2 & 7 & 9 \\
6 & 3 & 0 \\
\end{array}
\]

\[\text{ (units of $1000)}
\]

\[u:~ 2~ 4~ 0.5
\]

The unique optimal assignment is shown circled, the total gain being $16,000. One of the core vectors \((u, v)\) is also shown. The reader can verify that the matrix \((u + v)\) majorizes \((a_{ij})\), with equality only on the circled entries. The prices corresponding to this core vector are $22,000, $20,500, and $19,000 respectively. Given these prices, the first buyer clearly prefers the third house, as his gain is $2000 rather than $1000 or $1500. Similarly, the second buyer prefers the $4000 gain at house 1 to the gain of $3500 or $3000 at house 2 or 3. Finally, house 2 is the only one that the third buyer would even consider, at the stated prices.

The nonuniqueness of this solution is evident, since none of the buyers' comparisons were closer than $500, and only one seller (the third) is at, or even close to, his cost. Thus, any one price could be raised by $500 or either of the first two prices lowered by $500, without moving the outcome out of the core. Or all three prices could be increased together, in a market-wide price movement, until the "weakest" buyer (the third) is driven out.

The geometry of the core is shown in Fig. 2. Since optimality demands the unique set of assignments \(\{T^*, 2^*, 3^*\}\), we have depicted just the rectangular, 3-dimensional region in the (5-dimensional) imputation space in which the equations \(u_2 + v_2 = 8, u_3 + v_3 = 6, u_5 + v_1 = 2\) are satisfied. (The imputations outside this region cannot be attained without the aid of "third party" side payments.) The core is a five-sided polyhedron situated within this region, touching the boundaries \(u_3 = 0\) and \(v_3 = 0\). The solution point discussed above happens to lie in the bottom face, as shown by the open dot "*" in Fig. 3. The doubleheaded arrow indicates the direction of market-wide price changes. The low-price corner \((u^*, v^*)\), where all sellers get their minimum gains, is at the lower left; the high-price corner \((u^*, v^*)\) is at the upper right. The (Euclidean) distance between them is \(\sqrt{6}\) times $1000, or $2449; the distance measured in terms of the total effect on the sellers as a class is $4000.
4. A Special Case

4.1. The Horse Market of BÖHM-BEVERK

Discussions of price theory in economics have been enlivened by allusions to "Böhm-Bawerk's horses," the central figures in a microcosm market detailed in that author's Positive Theory of Capital, first published in 1891. Since this imaginary but numerically specific market model translates directly into an 18-person cooperative game, it makes an inviting target for the "big game hunter" anxious to try out his techniques. (Eighteen may be a small number in economics, but it is a large number in game theory!) Apart from the test of techniques, there is the prospect of an instructive confrontation among the various game solutions -- cores, stable sets, values, etc. -- and the classical solution as put forward by Böhm-Bawerk.

The basic data of the market are shown in Table 2. Each individual wishes to buy one horse. The horses themselves are all alike, but the traders have different "subjective valuations", ranging between $10 and $30, of what it is worth to own a horse. No restrictions are placed on communication, on transfers of money, or on transfers of horses. The basic game problem, informally stated, is to decide how the inherent profitability of the market, arising from the differences in subjective valuation, is to be shared among the horse-traders. This is also, of course, the basic economic problem.

<table>
<thead>
<tr>
<th>Sellers</th>
<th>Buyers</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1 values a horse at $10</td>
<td>B_1 values a horse at $30</td>
</tr>
<tr>
<td>A_2 values a horse at $11</td>
<td>B_2 values a horse at $12</td>
</tr>
<tr>
<td>A_3 values a horse at $15</td>
<td>B_3 values a horse at $26</td>
</tr>
<tr>
<td>A_4 values a horse at $17</td>
<td>B_4 values a horse at $24</td>
</tr>
<tr>
<td>A_5 values a horse at $20</td>
<td>B_5 values a horse at $22</td>
</tr>
<tr>
<td>A_6 values a horse at $21.50</td>
<td>B_6 values a horse at $21</td>
</tr>
<tr>
<td>A_7 values a horse at $25</td>
<td>B_7 values a horse at $20</td>
</tr>
<tr>
<td>A_8 values a horse at $26</td>
<td>B_8 values a horse at $18</td>
</tr>
<tr>
<td>A_9 values a horse at $26</td>
<td>B_9 values a horse at $17</td>
</tr>
<tr>
<td>A_9 values a horse at $26</td>
<td>B_10 values a horse at $15</td>
</tr>
</tbody>
</table>

4.2. The Characteristic Function

The reader will recognize that the present example, and the class of markets it exemplifies, is a decided simplification of our general model in one important respect: the absence of product differentiation. The matrix of input data (a_ij) can accordingly be reduced to a vector (h_j). As a result, the assignment matrix (a_i), now defined by

\[ a_{ij} = \max (0, h_j - c_i) \]

(4.1) enjoys the following special property: in each 2 \times 2 submatrix with nonzero entries, the sums of the diagonals are equal. This means that any set of assignments involving actual trades can be switched around, without loss or gain to anyone, since it only matters who buys and who sells, and not how they pair up. Hence the optimal assignment problem is rather trivial, and we shall be able to express the characteristic function in a very simple way.

Let S be an arbitrary coalition, and let its members be denoted by A_i,\ldots,A_n, B_{i1},\ldots,B_{km}, where the sellers A_i are arranged in order of increasing c_i and the buyers B_j in order of decreasing h_j. Then it is easy to see that

\[ v(S) = a_{i1} + a_{i2} + \ldots + a_{ik} \]

(4.2)

where k = min (l,m). In other words, for computational purposes we can assume that the "strongest" buyer in any coalition buys from the "strongest" seller, and so on down the line, until either the players of one type are exhausted or a pair is reached that cannot trade at a profit.

Applying these remarks to the data in Table 2, we find that

\[ v(M \cup N) = 20 + 17 + 11 + 7 + 2 + 0 + 0 + 0 = 57. \]

(4.3)

This is the total amount of profit inherent in the market, and it is achieved whenever the first five buyers acquire the horses of the first five sellers, and only then.

4.3. The Core

The core, too, is easy to determine, since the absence of product differentiation leads to a uniform market price. To see this let (u,v) be a typical imputation in the core. Suppose the amount of money received by some active seller i, which is \( u_i + c_i \), happens to be less than the amount of money paid by some active buyer j, which is \( h_j - v_j \). Then an improvement for \( ij \) would obviously be possible, since

\[ u_i + v_j < h_j - c_i = a_{ij}, \]

and \( (u,v) \) would not be in the core after all.

In short, each active seller must receive at least as much as each active buyer pays out. But no money enters the system; hence in any given core imputation all transactions take place at the same price.

Thus, the core can be described by means of a single parameter \( p \). A vector \( (u,v) \) is in the core if and only if, for \( p \) in some specified range, we have

\[ u_i = \max (0, p - c_i), \quad \forall i \in M, \]

\[ v_j = \max (0, h_j - p), \quad \forall j \in N. \]

(4.4)
The range over which $p$ can vary is given by the requirement that $\sum u_i + \sum v_j = v(M \cup N)$. This can only be satisfied if the same number of players of each type are active. more precisely, if the number of $i$ such that $p - c_i > 0$ does not exceed the number of $j$ such that $h_j - p \geq 0$, and vice versa. Geometrically the core is a straight line segment, and in its interior exactly the same number of traders of each type make a positive profit.

In the numerical example (Table 2), a price in the interval $21.00 < p < 21.50$ permits exactly five players of each type to trade at a profit. If the price were to exceed the upper limit, the sixth seller would put up his horse for sale and an imbalance would be created that would tend to drive the price down again. Similarly, a price less than $21 would find six (or more) buyers competing for five (or fewer) horses. The endpoints of the core are therefore

$$\{11.5, 10.5, 6.5, 4.5, 2.5, 0.5, 0, 0, 0, 0\}$$

and

$$\{11, 10, 6, 4, 1, 0, 0, 0; 9, 7, 5, 3, 1, 0, 0, 0, 0\}.$$ 

These are of course the extremal imputations $(u*, u*)$ and $(v*, v*)$ of Theorem 3 in Sec. 3.3.

4.4. Discussion

The core that we have determined in the preceding section is precisely the classical solution as given by BÖHM-BAWERK. A uniform market-wide price is established at a level which, though not precisely determined, is constrained within fairly narrow limits. If the consumers and suppliers in the market are sufficiently numerous, and if the tastes of the former and the costs of the latter are sufficiently variegated, then something close to a determinate outcome can be predicted. It is the classical balance of supply and demand. (See Sec. 5).

The one-dimensionality of the core represents the most extreme form of the elongation described in Theorem 3; this is a consequence of the lack of differentiation among the objects of trade. If a small "random" perturbation were applied to the model, giving the buyers slightly different subjective valuations for different horses, while the dimension of the core might jump from 1 to 5 it would nonetheless remain very slender. If the buyers should become more dissimilar in their tastes, however, the core would grow fatter. The ultimate in this direction would be a game in which each buyer has only one horse that he would accept at any price. Since the bargaining for each horse would then be independent of

the others, the game would decompose into smaller games, and the core would take the form of a rectangular parallelepiped (i.e., a product of intervals).

5. The Core and the Competitive Equilibrium

5.1. Convergence of the Core

An extensive literature now exists on the relation between the core of a market game and the notion of a competitive price equilibrium. Three basic features of this relationship are (1) that every competitive outcome is in the core, (2) that the core will generally contain other outcomes, and indeed may exist in situations where no competitive outcome exists, and (3) that under suitable conditions involving the number, size, or similarity of the economic agents, the core approaches or even coincides with the set of competitive outcomes.

The reader familiar with these ideas will already have recognized that in the assignment game every core outcome is competitive and vice versa. (See the discussion of prices in Sec. 3.2.) There is no doubt, then, as to the convergence of the two solution concepts; even if there are only a handful of players they coincide exactly. The reason for this lies in the special nature of the supply and demand functions (see Fig. 1), and the underlying preferences and indivisibilities that they reflect. The "lumpiness" of the commodity causes essentially the same nonuniqueness in the competitive prices as it does in the allocations that all coalitions will tolerate.

There is, however, another sort of convergence of interest, already suggested in Sec. 4.4. We shall describe it in general terms, having no actual theorems to offer. If the number of traders is increased, on both sides of the market, in such a way that their valuations for the products brought to market become more and more diverse (but remain bounded in a suitable sense), then the core will tend to shrink in size. The presence of a large variety of traders or products can in effect smooth out the supply and demand functions, and lead to more tightly determined competitive prices. In the BÖHM-BAWERK model, where there is no

1) Much of it, to be sure, has appeared since SHAPIRO [1955], our first treatment of the core of the assignment game. The basic paper of Demuynck and Scarf remains an excellent introduction; for a recent contribution to the subject (with a good bibliography) see Sato.

2) A final distribution of goods and money, or the corresponding vector of utilities, is called "competitive" if it can result from a behavioral process in which all decisions concerning production, trade, and consumption are made noncooperatively and in strict accordance with a set of prices that have been specified exogamously in such a way that the supply and demand of each commodity will be in balance.

3) More precisely, in the market models exemplified by houses or horses that we have been using to motivate the abstract assignment game. In the less specific "partnership" model of Sec. 2.3, the notion of competitive outcome must be reconstructed.

Incidentally, the indivisibilities and product differentiation in our model put it outside the scope of most of the general existence and convergence theorems in the literature.
product differentiation, this sharpening of the solution is a simple matter of narrowing the range of prices that will draw an equal number of buyers and sellers into the active market. In the more general model, however, the increasing dimensionality of the solution and the space in which it is defined makes a precise discussion of the shrinkage phenomenon more difficult.

5.2. Some Weaknesses

From both the economic and game-theoretic viewpoint, certain weaknesses in the core (= competitive) solution soon become apparent. For one thing, the core gives no consideration to the ability of individuals or coalitions to obstruct outcomes, though this is often a powerful lever in bargaining. For example, the coalition of all sellers can certainly block any trade, yet the core gives them no credit for this. Indeed, the inequality condition (3.6) for \( M = O \) (or for \( S = N \) for that matter) has no "bite" at all, and does not eliminate any individually rational outcomes from the solution. The core is based on what a coalition can do, not what it can prevent. The core is therefore not a really satisfactory basis for a bargaining theory.

There is a more subtle bargaining tactic, not of an obstructive nature, that also fails to be reflected in the core or competitive equilibrium. It can be illustrated in terms of the horse market of Table 2. Consider the "weak" players in the game — i.e., \( A_4 \), \( A_5 \), \( A_6 \) and \( B_1 - B_{10} \). They get nothing in the core. Although not technically dummies, they are quite unable to prevent the others from obtaining the full $57 profit, or to interfere in any other way with the imputations in the core. Nevertheless, the marginal weak player \( B_5 \) (for example) has an important role: it is his valuation of $21 that establishes the price floor in the core. Were he to quit the scene, the floor would drop to $20, and the sellers as a group would very likely suffer. By his very presence — i.e., his willingness to pay any amount up to $21 for a horse — he performs a service to some of the players (and a disservice to others). What is it worth?

A similar capability exists for the marginal weak seller \( A_6 \), who sets the ceiling of $21.50 in the solution. In lesser measure, the other weak players also have some bargaining power of this kind. The seventh buyer \( B_7 \), for example, might argue that his participation in the bidding tends to limit the sixth buyer's "leverage", since if \( B_7 \) should quit the scene, then \( B_6 \) might be able to command a larger bribe from the active sellers for remaining on hand to protect the $21 floor.

References


1) If we were so willing, some Pareto optimality might reassert itself, in the guise of a counter-offer by the sellers, and induce \( B_5 \) to return to the market to his further profit.
2) An instructive critique of the classical solution will be found in von NEUMANN and Morgenstern loc. cit.; see also SHAPIRO [1961] and Sec. 3 of SHAPIRO and SHUBIK [1969].

1) Even in the BÖHM-BAXTER case, if the price range does not narrow fast enough, the (euclidean) distance between the endpoints of the core may actually increase as the price interval goes to zero.
2) "The core is sometimes described as "the set of outcomes that no coalition can block"; this unfortunate and misleading description stems from a counter-intuitive use of the term "block" in the mathematical terminology used by some writers. A better rendering is "the set of outcomes that no coalition can improve upon".
3) As shown by the fact that \( v(A) = 0 \).
Possible Conflict Structure of the European Security Conference

By UTZ-PETER REICH 1)

Abstract: A group of 31 West German political scientists has played a European Security conference. Eight nations were represented, FRG, GDR, USA, USSR, France, UK, Poland, CSSR. In the course of the play 26 statements were valued under the aspect of incorporating them into a conference-communicating to be accepted by the conference. In the article, these values are processed into a game-theoretical analysis of the conflict played. Starting from the solution of the bargaining game, symmetry, dynamics, stability and cooperability of the game are defined and investigated. The result may be taken as a possible structure of the conflict as it may develop at the official European Security Conference.

I. Introduction

On November 7th—9th 1969, a group of 31 political scientists from all over West Germany met at Haus Rissen, Hamburg, to play a European Security Conference. The game was designed not to simulate the actual conference but to study the underlying conflict of interests. The degree to which the play is relevant to the actual conference was recognized as problematic. In this report, no reference will be made to this problem. Only the experiment itself will be described and the data gathered will be analyzed in terms of game-theory.

The intention of this report in general is to refrain from political interpretation but to give an example of how game-theory can be used in actual political problems.

II. The Play

The participants, Instructors at University Political Science Departments, and Assistants at research institutes, had been assigned in advance to one of the following countries: FRG, GDR, US, USSR, UK, France, CSSR, Poland. The assignment had followed as far as possible the choices of the participants. In addition, the participants had been requested to outline a policy, which they intended to follow at the conference and communicate it to their “fellow countrymen”. Half of the participants had responded.

The first meeting on Friday night was spent introducing the game and practicing. After that only little time was left for discussing the setting of the play. The vast majority voted that a European Security Conference would not be possible in 1970, and agreed almost unanimously that it was possible later. The discussion about what developments would have to take place for the conference

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